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A NOTE ON THE SOLUTION FOR TWO ASYMMETRIC BOUNDARY VALUE PROBLEMS

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Abstract—The solution is given for two mixed-mixed boundary value problems in classical elasticity theory. The first is a rigid disk, imbedded in an infinite space and given a constant displacement parallel to its plane. The second is a problem exterior to the first: two half-spaces are joined to an infinitesimally thin, rigid sheet containing a circular hole, the half-spaces being joined through the hole. The sheet is caused to move.

1. INTRODUCTION

CERTAIN boundary value problems in classical elasticity theory have been written in terms of harmonic functions and their solution obtained by means of potential theory. Solutions are given in Green and Zerna [1] for boundary value problems of a half-space with no tangential shearing forces. The two problems considered there were of axial symmetry: a rigid punch indenting a half-space and the penny-shaped crack. Other axially symmetric problems were considered by Collins [2] and Goodman [3] in which a radial shear only is applied. Mindlin [4] solves the asymmetric problem of tangential contact between two identical spheres through the use of potential functions that eliminate stresses normal to the half-space and shear stresses tangential to one of the coordinate axes in the plane of the half-space.

It is the purpose of this note to consider two solutions for the case of zero displacement normal to the plane of the half-space. The following two additional restrictions are to be separately considered. The first is zero tangential shear and the second is zero displacement, each in the direction of one of the coordinate axes.

For the work that follows a Cartesian coordinate system (x, y, z) will be used. The displacements corresponding to these coordinates are given by (u_x, u_y, u_z) and the stresses are given by $(\sigma_{zx}, \sigma_{zy}, \sigma_{zz})$, where the stresses are computed on an element whose normal is parallel to the z-axis. In addition it will prove convenient to use a cylindrical coordinate system (r, θ, z) whose center and z-axis coincide with the center and z-axis of the Cartesian system of coordinates. We choose the following displacement solutions to the field equations of elasticity:*

$$2\mu u_x = -(3-4\nu)\,\partial\gamma/\partial z + x\,\partial^2\gamma/\partial x\partial z + z\,\partial\psi_z/\partial x + \partial\Phi/\partial x, \tag{1}$$

$$2\mu u_y = x\,\partial^2\gamma/\partial y\partial z + z\,\partial\psi_z/\partial y + \partial\Phi/\partial y, \tag{2}$$

$$2\mu u_z = -(3-4\nu)\,\psi_z + x\,\partial^2\gamma/\partial z^2 + z\,\partial\psi_z/\partial z + \partial\Phi/\partial z, \tag{3}$$

* See, for example, Green and Zerna [1], pp. 169–170.

where $\nabla^2\gamma = \nabla^2\psi_z = \nabla^2\Phi = 0$. Substitution of equations (1)–(3) into the stress–displacement relations for isotropic, elastic solids under isothermal conditions gives the following values for the stresses:

$$\sigma_{zz} = -2(1-\nu)\,\partial\psi_z/\partial z - 2\nu\,\partial^2\gamma/\partial x\partial z + x\,\partial^3\gamma/\partial z^3 + z\,\partial^2\psi_z/\partial z^2 + \partial^2\Phi/\partial z^2, \tag{4}$$

$$\sigma_{zx} = -(1-2\nu)(\partial\psi_z/\partial x + \partial^2\gamma/\partial z^2) + \partial^2\Phi/\partial x\partial z + z\,\partial^2\psi_z/\partial x\partial z + x\,\partial^3\gamma/\partial x\partial z^2, \tag{5}$$

$$\sigma_{zy} = -(1-2\nu)\,\partial\psi_z/\partial y + \partial^2\Phi/\partial y\partial z + z\,\partial^2\psi_z/\partial y\partial z + x\,\partial^3\gamma/\partial y\partial z^2. \tag{6}$$

The solutions to follow will consider special cases of the above equations for stresses and displacements.

2. DISPLACEMENT OF RIGID DISK

We consider in this Section the problem of a rigid disk imbedded in an infinite, elastic solid with the plane of the disk coinciding with the (x, y)-plane of the coordinate

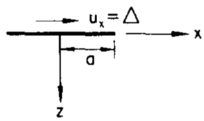

FIG. 1(a).

system given above (Fig. 1(a)). The center and axis of the disk coincide with the center and z-axis of the coordinate system. Boundary conditions for this problem are the following:

$$u_z = u_y = 0, \qquad z = 0, \qquad (0 \le r \le a), \tag{7}$$

$$u_x = \Delta, \qquad z = 0, \qquad (0 \le r \le a), \tag{8}$$

$$u_z = \sigma_{zx} = \sigma_{zy} = 0, \qquad z = 0, \qquad (a < r < \infty), \tag{9}$$

To satisfy these conditions we consider the particular case of (1)–(6) when $u_z = \sigma_{zy} = 0$ on the plane $z = 0$. To satisfy these conditions it is sufficient that the following two potentials be equal to zero:

$$\psi_z = 0, \tag{10}$$

$$\partial\Phi/\partial z + x\,\partial^2\gamma/\partial z^2 - z\,\partial^2\gamma/\partial x\partial z = 0. \tag{11}$$

We now take the potential in the form given by Green and Zerna*

$$\partial\gamma/\partial z = \tfrac{1}{2}\int_{-a}^{a} f(t)R^{-1}\,dt, \qquad f(t) = f(-t), \tag{12}$$

where $R = [r^2+(z+it)^2]^{\frac{1}{2}}$.

We note the following results for R as $z \to 0$:

$$\begin{aligned}\lim_{z\to 0}\,[r^2+(z+it)^2]^{\frac{1}{2}} &= \lim_{z\to 0}\,[r^2+(z-it)^2]^{\frac{1}{2}} = (r^2-t^2)^{\frac{1}{2}} \qquad (t < r),\\ \lim_{z\to 0}\,[r^2+(z+it)^2]^{\frac{1}{2}} &= -\lim_{z\to 0}\,[r^2+(z-it)^2]^{\frac{1}{2}} = i(t^2-r^2)^{\frac{1}{2}} \qquad (r < t),\end{aligned} \tag{13}$$

* *Loc. cit.* pp. 173–174.

and the values for $\partial\gamma/\partial z$ and $\partial^2\gamma/\partial z^2$ as $z \to 0$:

$$\begin{aligned}\partial\gamma/\partial z &= \lim_{z\to 0} \tfrac{1}{2}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t \\ &= \int_0^r f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (0 \le r \le a) \\ &= \int_0^a f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (a < r < \infty),\end{aligned} \tag{14}$$

$$\begin{aligned}\partial^2\gamma/\partial z^2 &= \lim_{z\to 0} \frac{1}{2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_{-a}^{a} (z+it)f(t)R^{-1}\,\mathrm{d}t \\ &= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a tf(t)(t^2-r^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (0 \le r \le a).\end{aligned} \tag{15}$$

Using the definitions as given in equations (11), (12) and (15) we obtain the following result for $\partial\Phi/\partial z$:

$$\frac{\partial\Phi}{\partial z} = \frac{\cos\theta}{2i}\frac{\mathrm{d}}{\mathrm{d}r}\int_{-a}^{a} tf(t)R^{-1}\,\mathrm{d}t. \tag{16}$$

We find that Φ and $\partial\Phi/\partial z$ have the following value on $z = 0$:

$$\begin{aligned}\Phi &= -\frac{\cos\theta}{r}\int_0^r t^2 f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (0 \le r \le a) \\ &= -\frac{\cos\theta}{r}\int_0^a t^2 f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (a < r < \infty),\end{aligned} \tag{17}$$

$$\begin{aligned}\partial\Phi/\partial z &= -\cos\theta\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a tf(t)(t^2-r^2)^{-\frac{1}{2}}\,\mathrm{d}t \qquad (0 \le r \le a) \\ &= 0 \qquad (a < r < \infty).\end{aligned} \tag{18}$$

Using the preceding results, we can put the stresses and displacements in terms of the one unknown function, $f(t)$ as follows:

$$\begin{aligned}2\mu u_x &= -(3-4\nu)\frac{1}{2}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t + \frac{x}{2}\frac{\partial}{\partial x}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t + \frac{\partial}{\partial x}\frac{x}{2i}\int_{-a}^{a} tf(t)(z+it+R)^{-1}R^{-1}\,\mathrm{d}t, \\ 2\mu u_y &= \frac{x}{2}\frac{\partial}{\partial y}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t + \frac{\partial}{\partial y}\frac{x}{2i}\int_{-a}^{a} tf(t)(z+it+R)^{-1}R^{-1}\,\mathrm{d}t, \\ 2\mu u_z &= \frac{z}{2}\frac{\partial}{\partial x}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t,\end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_{zz} &= (1-2\nu)\frac{1}{2}\frac{\partial}{\partial x}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t+\frac{z}{2}\frac{\partial}{\partial x}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_{-a}^{a}(z+it)f(t)R^{-1}\,\mathrm{d}t,\\ \sigma_{zx} &= -(1-\nu)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_{-a}^{a}(z+it)f(t)R^{-1}\,\mathrm{d}t+\frac{z}{2}\frac{\partial^2}{\partial x^2}\int_{-a}^{a} f(t)R^{-1}\,\mathrm{d}t.\end{aligned} \tag{20}$$

On $z = 0$ $(0 \le r \le a)$ the results become the following:

$$\left\{\begin{aligned} u_y &= u_z = 0,\\ 2\mu u_x &= -(3-4\nu)\int_0^r f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t+x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^r f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t\\ &\quad -\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos\theta}{r}\int_0^r t^2 f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t\end{aligned}\right. \tag{21}$$

$$\left\{\begin{aligned} \sigma_{zy} &= 0,\\ \sigma_{zx} &= -2(1-\nu)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a tf(t)(t^2-r^2)^{-\frac{1}{2}}\,\mathrm{d}t,\\ \sigma_{zz} &= (1-2\nu)\cos\theta\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r f(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t.\end{aligned}\right. \tag{22}$$

For $r > a$ the two shear stresses vanish on $z = 0$. If the function, $f(t)$, is taken to be constant, then it is seen the boundary conditions of the problem are satisfied. In particular if $f(t) = -8\mu\Delta/\pi(7-8\mu)$ then the stresses and displacements computed on $z = 0$ are

$$\left\{\begin{aligned} 2\mu u_x &= 2\mu\Delta \\ &= \frac{4}{\pi}\mu\Delta\sin^{-1}\left(\frac{a}{r}\right)+\frac{4\mu\Delta}{\pi(7-8\nu)}ar^{-2}(r^2-a^2)^{\frac{1}{2}}\cos 2\theta \qquad (a < r < \infty),\\ 2\mu u_y &= 0 \qquad (0 \le r \le a)\\ &= \frac{4\mu\Delta}{\pi(7-8\nu)}ar^{-2}(r^2-a^2)^{\frac{1}{2}}\sin 2\theta \qquad (a < r < \infty),\end{aligned}\right. \tag{23}$$

$$\left\{\begin{aligned} \sigma_{zx} &= -\frac{16(1-\nu)}{\pi(7-8\nu)}\mu\Delta(a^2-r^2)^{-\frac{1}{2}} \qquad (0 \le r \le a),\\ \sigma_{zz} &= 0 \qquad (0 \le r \le a)\\ &= \frac{8(1-2\nu)(1-\nu)}{\pi(7-8\nu)}\mu\Delta\frac{a}{r}(r^2-a^2)^{-\frac{1}{2}} \qquad (a < r < \infty),\end{aligned}\right. \tag{24}$$

3. DISPLACEMENT OF RIGID SHEET

In this Section the analogous problem of a rigid sheet imbedded in an infinite elastic solid is considered. The sheet has a circular hole in its center whose axis coincides with the z-axis while the plane of the sheet coincides with the (x, y)-axis (Fig. 1(b)). The sheet

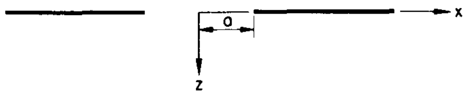

FIG. 1(b).

is subsequently given a constant displacement parallel to the x-axis causing shear stresses $\sigma_{zx} = \pm\sigma_0$ as $z \to \pm\infty$. Boundary conditions that are equivalent to the above are:

$$\begin{cases} \sigma_{zy} = 0 \\ \sigma_{zx} = \sigma_0 \end{cases} \quad z = 0 \quad (0 \le r \le a), \tag{25}$$

$$u_x = u_y = 0 \quad z = 0 \quad (a < r < \infty), \tag{26}$$

$$u_z = 0 \quad z = 0. \tag{27}$$

To solve this problem we consider the case where displacements u_y and u_z vanish on $z = 0$. From equations (2) and (3) we see that these conditions can be satisfied if the following potential functions are set equal to zero:

$$V_1 = -(3-4\nu)\psi_z + x\,\partial^2\gamma/\partial z^2 - z\,\partial^2\gamma/\partial x\,\partial z + \partial\Phi/\partial z, \tag{28}$$

$$V_2 = x\,\partial\gamma/\partial z - z\,\partial\gamma/\partial x + \Phi. \tag{29}$$

From (28) and (29) Φ and ψ_z may be written in terms of γ as follows:

$$\Phi = z\,\partial\gamma/\partial x - x\,\partial\gamma/\partial z, \tag{30}$$

$$\cdot(3-4\nu)\psi_z = \partial\gamma/\partial x. \tag{31}$$

Using equations (30) and (31) we obtain the representation of displacement and stresses in terms of γ below:

$$\begin{cases} 2\mu u_x = -4(1-\nu)\,\partial\gamma/\partial z + \dfrac{4(1-\nu)}{(3-4\nu)} z\,\partial^2\gamma/\partial x^2, \\ 2\mu u_y = \dfrac{4(1-\nu)}{(3-4\nu)} z\,\partial^2\gamma/\partial x\partial y, \\ 2\mu u_z = \dfrac{4(1-\nu)}{(3-4\nu)} z\,\partial^2\gamma/\partial x\,\partial z, \end{cases} \tag{32}$$

$$\begin{cases} \sigma_{zz} = \dfrac{4(1-\nu)}{(3-4\nu)}(1-2\nu)\dfrac{\partial^2\gamma}{\partial x\,\partial z} + \dfrac{4(1-\nu)}{(3-4\nu)} z \dfrac{\partial^3\gamma}{\partial x\,\partial z^2}, \\ \sigma_{zy} = \dfrac{2(1-\nu)}{(3-4\nu)}\dfrac{\partial^2\gamma}{\partial x\,\partial y} + \dfrac{4(1-\nu)}{(3-4\nu)} z \dfrac{\partial^3\gamma}{\partial x\,\partial y\,\partial z}, \\ \sigma_{zx} = \dfrac{2(1-\nu)}{(3-4\nu)}\dfrac{\partial^2\gamma}{\partial x^2} - 2(1-\nu)\dfrac{\partial^2\gamma}{\partial z^2} + \dfrac{4(1-\nu)}{(3-4\nu)} z \dfrac{\partial^3\gamma}{\partial x^2\,\partial z}. \end{cases} \tag{33}$$

On $z = 0$ the displacements and stresses become

$$\begin{cases} 2\mu u_x = -4(1-\nu)\,\partial\gamma/\partial z, \\ u_y = u_z = 0, \end{cases} \tag{34}$$

$$\begin{cases} \sigma_{zz} = \dfrac{4(1-\nu)(1-2\nu)}{(3-4\nu)}\,\partial^2\gamma/\partial x\,\partial z, \\ \sigma_{zy} = \dfrac{2(1-\nu)}{(3-4\nu)}\,\partial^2\gamma/\partial x\,\partial y, \\ \sigma_{zx} = \dfrac{2(1-\nu)}{(3-4\nu)}\,\partial^2\gamma/\partial x^2 - 2(1-\nu)\,\partial^2\gamma/\partial z^2, \end{cases} \tag{35}$$

and the problem is reduced to finding a function γ that satisfies the boundary conditions. We take the function $\partial\gamma/\partial z$ to be that given by Collins [2] for penny-shaped cracks under axially symmetric tension perpendicular to the axis of the crack. This function is

$$\partial\gamma/\partial z = \frac{1}{2i}\int_{-a}^{a} g(t)R^{-1}\,\mathrm{d}t, \qquad g(t) = -g(-t). \tag{36}$$

Using the results of (13), we obtain the following values of the function itself and certain derivatives on $z = 0\,(0 \le r \le a)$

$$\partial\gamma/\partial z = -\int_r^a g(t)(t^2-r^2)^{-\frac{1}{2}}\,\mathrm{d}t, \tag{37}$$

$$\partial^2\gamma/\partial z^2 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r tg(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t, \tag{38}$$

$$\partial\gamma/\partial x = -\frac{\cos\theta}{r}\int_0^r tg(t)(r^2-t^2)^{-\frac{1}{2}}\,\mathrm{d}t. \tag{39}$$

From the form of these functions it can be seen that the boundary conditions for the problem given by equations (25) and (26) can be satisfied on the surface $z = 0$ if the following value is given for $f(t)$:

$$f(t) = -\frac{2}{\pi}\frac{(3-4\nu)\sigma_0}{(1-\nu)(7-8\nu)}t. \tag{40}$$

The resulting values for the displacements and stresses on $z = 0$ are given below:

$$\left\{\begin{aligned} u_y &= u_z = 0, \\ 2\mu u_x &= -\frac{8}{\pi}\frac{(3-4\nu)}{(7-8\nu)}(a^2-r^2)^{\frac{1}{2}} && (0 \le r \le a) \\ &= 0 && (a < r < \infty), \end{aligned}\right. \tag{41}$$

$$\left\{\begin{aligned} \sigma_{zz} &= -\frac{8}{\pi}\frac{(1-2\nu)}{(7-8\nu)}\sigma_0 r(a^2-r^2)^{\frac{1}{2}} && (0 \le r \le a) \\ &= 0 && (a < r < \infty), \\ \sigma_{zx} &= \sigma_0 && (0 \le r \le a) \\ &= \frac{2}{\pi}\sigma_0\left[\sin^{-1}\left(\frac{a}{r}\right) - a(r^2-a^2)^{-\frac{1}{2}}\right] \\ &\quad + \frac{2}{\pi}\frac{\sigma_0}{(7-8\nu)}\left[ar^{-2}(r^2-a^2)^{\frac{1}{2}} - a(r^2-a^2)^{-\frac{1}{2}}\right]\cos 2\theta && (a < r < \infty), \\ \sigma_{zy} &= 0 && (0 \le r \le a) \\ &= \frac{2}{\pi}\frac{\sigma_0}{(7-8\nu)}\left[a(r^2-a^2)^{-\frac{1}{2}} - ar^{-2}(r^2-a^2)^{\frac{1}{2}}\right]\sin 2\theta && (a < r < \infty). \end{aligned}\right. \tag{42}$$

4. CONCLUSIONS

We observe the following features of the solutions given here. An interesting result is that the first problem can be solved by assuming that the shear stresses perpendicular to the motion of the disk vanish everywhere on $z = 0$, and that on the disk itself shear stresses act only in the direction of the movement. As is expected, stress singularities in the shear stress occur at all points on the edge of the disk, $r = a^-$; stress singularities in the normal stress occur exterior to the disk, $r = a^+$. For the second problem the only tangential displacement present is in the direction of the only tangential stress interior to $r = a$ on $z = 0$. Singularities in the normal stress occur at points where $r = a^-$ and singularities in the shear stresses occur when $r = a^+$.

Westmann [5] obtains essentially similar results for interior and exterior crack problems. For two half-spaces joined by a circular disk and given constant displacement at $z = \infty$ he shows that the stresses perpendicular to the direction of the displacement vanish on the contact surface. He shows for a penny-shaped crack under uniform shear that all displacements perpendicular to the direction of the applied shear stress vanish in the plane of the crack.

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Zusammenfassung—Für zwei gemischte Gemischtgrenzwertprobleme der klassischen Elastizitätstheorie werden die Lösungen angegeben. Das erste Problem befasst sich mit einer in einen grenzenlosen Raum eingebetteten, steifen Scheibe, welche in einer parallel zu ihrer Ebene liegenden Richtung verschoben wird. Das zweite Problem liegt ausserhalb des ersten Problemes: Zwei Halbräume liegen einer unendlich dünnen, steifen Fläche an. Diese enthält ein rundes Loch, durch welches die Halbräume miteinander in Verbindung stehen. Die Fläche wird in Bewegung gesetzt.

Абстракт—Дано решение для двух проблем, со смешанным граничным значением, в классической теории упругости. Первая проблема: жесткий диск, погруженный в бесконечное пространство, имеет постоянное смещение, параллельное его плоскости. Вторая проблема является внешней по отношению к первой: два полупространства присоединены к бесконечно тонкому жесткому листу, имеющему круглое отверстие, через которое эти два полупространства соединены. Лист подвергнут перемещению.